WPR1 – Practice Problems – Sample Solutions

1) $y'(x) = \sec^2(\sin x)\cdot\cos(x)$

2) $f'(t) = e^{t\cos t}\cdot(-t\sin t + \cos t)$

3) $y'(t) = 2^{\sin(\pi x)}\cdot(\cos(\pi x)\cdot\pi)\cdot\ln 2$

4) $f'(x) = \frac{1}{2}(6x+1)^{-1/2}\cdot 6$

5) $f'(x) = e^{5x+1}\cdot 5$

6) $f'(t) = \sec^2(5x^2+3)\cdot 10x$

7) $p'(\theta) = 3^{\theta^2+1}\cdot(2\theta)\cdot\ln 3$

8) $f'(x) = \cos(2x)\cdot 2$

9) $g'(x) = e^{-2x^2}\cdot(-4x)$

10) $f'(x) = 2(5x^2+1)(10x)$

11) $y'(x) = -\sin(\frac{3}{2}\pi t)\cdot(\frac{3}{2}\pi)$

12) $g'(x) = \dfrac{(\sqrt{x^2+2})(3) - (3x)\left(\frac{1}{2}(x^2+2)^{-1/2}(2x)\right)}{x^2+2}$

13) $f'(x) = e^{\cot(x)}(-\csc^2(x))$

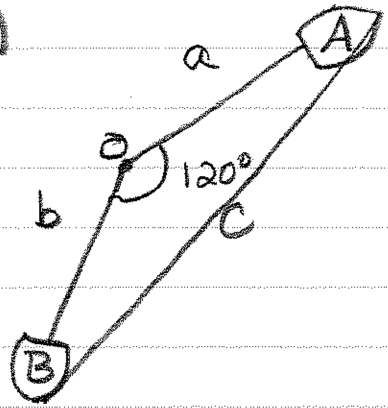
14) $f'(x) = x e^{-x^2}(-2x) + e^{-x^2}$

15) $f'(x) = \dfrac{(x^2-1)(1) - x(2x)}{(x^2-1)^2}$ $\quad f'(x) = x(-(x^2-1)^{-2}(2x)) + (x^2-1)^{-1}$

[These derivatives are the same!]

WPR1 - Practice Problems - Sample Solutions

16)



a = distance from O to ship A, VAR WRT t
b = " " " " " B, VAR WRT t
c = distance between A + B, VAR WRT t

Given $\frac{da}{dt} = 20$ mi/hr, $\frac{db}{dt} = 30$ mi/hr

What is $\frac{dc}{dt}$ when $a = 8$ mi + $b = 6$ mi?

From the Law of cosines:

$$c^2 = a^2 + b^2 - 2ab\cos(120^\circ)$$

This is in degrees so use DEG on your calculator.

$$2c\frac{dc}{dt} = 2a\frac{da}{dt} + 2b\frac{db}{dt} - 2a\cos(120^\circ)\frac{db}{dt} - 2b\cos(120^\circ)\frac{da}{dt}$$

$$\frac{dc}{dt} = \frac{a}{c}\frac{da}{dt} + \frac{b}{c}\frac{db}{dt} - \frac{a}{c}\cos(120^\circ)\frac{db}{dt} - \frac{b}{c}\cos(120^\circ)\frac{da}{dt}$$

Need c when $a = 8$ mi, $b = 6$ mi:

$$c^2 = (8)^2 + (6)^2 - 2(8)(6)\cos(120^\circ) = 100 - 96\left(-\tfrac{1}{2}\right)$$

$$c^2 = 148 \Rightarrow c = \pm\sqrt{148},\ c = \sqrt{148} \approx 12.166$$

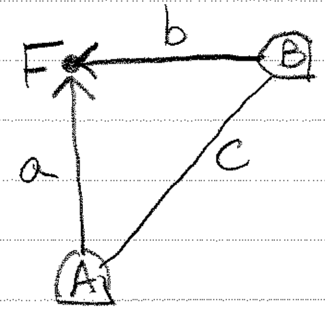
$$\frac{dc}{dt} = \frac{8}{\sqrt{148}}(20) + \frac{6}{\sqrt{148}}(30) - \frac{8}{\sqrt{148}}\left(-\tfrac{1}{2}\right)(30) - \frac{6}{\sqrt{148}}\left(-\tfrac{1}{2}\right)(20)$$

$$= \frac{160}{\sqrt{148}} + \frac{180}{\sqrt{148}} + \frac{120}{\sqrt{148}} + \frac{60}{\sqrt{148}} = \frac{520}{\sqrt{148}} \approx 42.74$$

$$\frac{dc}{dt} \approx 42.74 \text{ mi/hr.}$$

The distance between them is changing at the rate of ≈ 42.74 mi/hr

17



a = distance from boat A to Finish, VAR
b = " " " B " " , VAR
c = distance between boats, VAR

Given $\frac{da}{dt} = -13\text{mph}$ and $\frac{db}{dt}$ = constant.

When $a=b$ we are told $c = 16\text{mi}$ and $\frac{dc}{dt} = -17\text{mph}$. Since the boats' speed is constant, and we're told that at some point they are at the same distance from the finish line, we need to compare the speeds of the 2 boats to decide who wins. We're already given $\frac{da}{dt}$ so we just need $\frac{db}{dt}$.

$$a^2 + b^2 = c^2$$

$$\frac{d}{dt}(a^2+b^2) = \frac{d}{dt}(c^2)$$

$$2a\frac{da}{dt} + 2b\frac{db}{dt} = 2c\frac{dc}{dt}$$

$$\Rightarrow \underline{\frac{db}{dt}} = \frac{c}{b}\frac{dc}{dt} - \overset{1}{\frac{a}{b}}\frac{da}{dt}$$

Find b when $a=b$ and $c = 16\text{mi}$:

$$2b^2 = 16^2$$

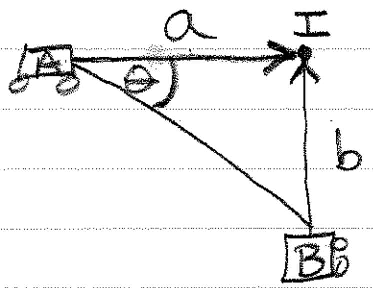
$$b^2 = \frac{16^2}{2} \Rightarrow b = \frac{1}{\sqrt{2}}16\text{mi} = a$$

$$\frac{db}{dt} = \frac{16\text{mi}}{\frac{1}{\sqrt{2}}16\text{mi}}(-17\text{mph}) - (1)(-13\text{mph}) = -17\sqrt{2}\text{mph} + 13\text{mph}$$

$$\frac{db}{dt} \approx -11.04\text{ mph}$$

$\boxed{\text{Boat a is traveling faster than boat b, so Boat a wins.}}$

18)



a = distance from car A to intersection I, VAR
b = " " " B " " VAR
$\theta = \angle IAB$, VAR WRT t

Given: $\frac{da}{dt} = -30$ mph, $\frac{db}{dt} = -22.5$ mph
Find $\frac{d\theta}{dt}$ when $a = 300$ ft and $b = 400$ ft.

Note: These rates are negative because of the way I defined a and b. The cars are approaching I so the distances are decreasing with time. Hence $\frac{da}{dt}$ + $\frac{db}{dt}$ are negative

$$\tan\theta = \frac{b}{a}$$

$$\frac{d}{dt}(\tan\theta) = \frac{d}{dt}\left(\frac{b}{a}\right)$$

$$* \quad \sec^2\theta \frac{d\theta}{dt} = \frac{a\frac{db}{dt} - b\frac{da}{dt}}{a^2}$$

Need $\sec\theta$! (There is more than one way to do this.)

$$\sec\theta = \frac{1}{\cos\theta} = \frac{\sqrt{a^2+b^2}}{a}$$

$$\therefore \sec^2\theta = \frac{a^2+b^2}{a^2}$$

Substituting this back into $*$:

$$\left(\frac{a^2+b^2}{a^2}\right)\frac{d\theta}{dt} = \frac{a\frac{db}{dt} - b\frac{da}{dt}}{a^2}$$

$$\Rightarrow \frac{d\theta}{dt} = \frac{a\frac{db}{dt} - b\frac{da}{dt}}{a^2+b^2}$$

Note: Need to convert mph to ft/hr or ft to miles... use $\frac{1\text{mi}}{5280\text{ft}}$:

$a = 300\text{ft} \cdot \frac{1\text{mi}}{5280\text{ft}} \doteq .05682\text{mi}$

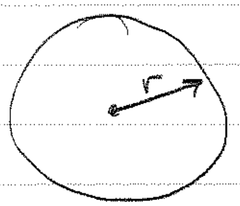
$b = 400\text{ft} \cdot \frac{1\text{mi}}{5280\text{ft}} \doteq .07576\text{mi}$

$$\frac{d\theta}{dt} = \frac{(.05682\text{mi})(-22.5\text{mi/hr}) - (.07576\text{mi})(-30\frac{\text{mi}}{\text{hr}})}{(.05682\text{mi})^2 + (.07576\text{mi})^2}$$

$$= \frac{.99435\ \text{mi}^2/\text{hr}}{.0089681\ \text{mi}^2} = 110.88\ \frac{\text{rads}}{\text{hr}}$$

$$\boxed{\frac{d\theta}{dt} = 110.88\ \frac{\text{radians}}{\text{hr}} = 1.848\ \frac{\text{radians}}{\text{min}} = .0308\ \frac{\text{radians}}{\text{second}}}$$

20)



r = radius of circle, VAR WRT t
A = area of circle, VAR WRT t

Find $\frac{dA}{dt}$ in terms of $\frac{dr}{dt}$.

$A = \pi r^2$

$\frac{d}{dt}(A) = \frac{d}{dt}(\pi r^2)$

$$\boxed{\frac{dA}{dt} = 2\pi r \frac{dr}{dt}}$$

Given: $\frac{dr}{dt} = 2$ cm/s
Find how fast the area is increasing when $r = 8$cm?

$$\frac{dA}{dt} = 2\pi(8\text{cm})(2\text{ cm/s}) = 32\pi \text{ cm}^2/\text{s} \approx 100.5 \text{ cm}^2/\text{s}$$

$\boxed{\text{The area of the spill is increasing at } 32\pi \text{ cm}^2/\text{s} \approx 100.5 \text{cm}^2/\text{s}}$

21) (Same setup as in problem 20.) Given: $\frac{dA}{dt} = 6\text{mi}^2/\text{hr}$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

Find $\frac{dr}{dt}$ when $A = 9\text{mi}^2$.

$$\Rightarrow \frac{dr}{dt} = \frac{1}{2\pi r} \frac{dA}{dt}.$$

Since $A = 9\text{mi}^2$, this means $\pi r^2 = 9\text{mi}^2$, i.e. $r = \pm\sqrt{\frac{9\text{mi}^2}{\pi}}$.
So, find $\frac{dr}{dt}$ when $r = \frac{3\text{ mi}}{\sqrt{\pi}}$.

$$\frac{dr}{dt} = \frac{1}{2\pi\left(\frac{3\text{mi}}{\sqrt{\pi}}\right)} \cdot 6\text{mi}^2/\text{hr} = \frac{1}{\sqrt{\pi}} \frac{\text{mi}}{\text{hr}} \approx .564 \text{ mi/hr}.$$

$\boxed{\text{The radius of the spill increases at the rate of } 0.564 \text{ mi/hr.}}$

27) The MVT says if f is a differentiable function on an interval $[a,b]$, then there exists a number c $(a<c<b)$ such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}.$$

Ms. Fastdriver should get a citation if her instantaneous velocity is over the speedlimit.

Her average velocity is $\frac{(113.8-1.2)\text{mi}}{(2{:}50\text{pm}-1{:}30\text{pm})}=\frac{112.6\text{mi}}{4/3\text{ hr}}$

$=84.45\text{mph}$

By the MVT, Ms. Fastdriver's instantaneous velocity must have been equal to 84.45mph at some time during that trip. This is clearly over the speed limit, so NJ Transit should issue a ticket.

28) By the MVT, there is some time t_* so that

$$v'(t_*)=\frac{v(b)-v(a)}{b-a} \quad \text{where } a=2{:}00\text{pm} + b=2{:}10\text{pm},$$

i.e. $v'(t_*)=\dfrac{50\text{mph}-30\text{mph}}{10\text{min}\cdot\frac{1\text{hr}}{60\text{min}}}=\dfrac{20\text{ mi/h}}{1/6\text{ h}}=120\text{mi/hr}^2.$

So by the MVT there is some time t_* between 2:00pm and 2:10pm so that the acceleration is 120 mi/hr^2 at the time t_*.