Solutions to Optional Practice for TEE

1) $f(x) = ax^2 + bx + c$, $f'(x) = 2ax + b$, $f''(x) = 2a$ $\Rightarrow$

$f(3) = a9 + b3 + c = -46$ (3)

$f'(3) = 2a(3) + b = 2$ (2)

$f''(3) = 2a = 6$ (1)

(1) $2a = 6 \Rightarrow \underline{a = 3}$

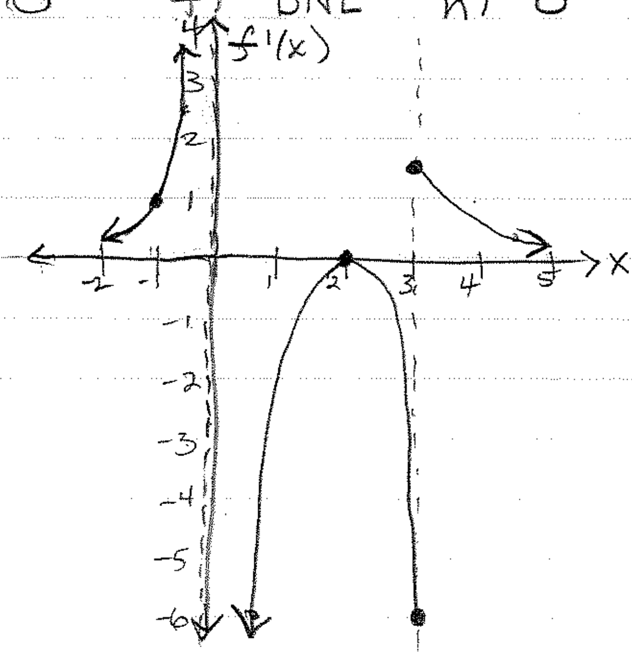
Substituting into (2) yields $6(3) + b = 2$
$\Rightarrow b = 2 - 18$
$\underline{b = -16}$

Substituting $a = 3$ and $b = -16$ into (3) yields $(3)(9) - 16(3) + c = -46$
$27 - 48 + c = -46$
$c = -46 + 48 - 27$
$c = -25$

$\boxed{\text{The polynomial is } f(x) = 3x^2 - 16x - 25.}$

2) (Note for a–c: The limits as x approaches 0 are similar to top p. 128 and Example 8 p. 104. If you put "DNE" where I wrote "∞" in a), b), c), you are also correct based on what we've learned about limits.)

a) ∞
b) ∞
c) ∞
d) 0
e) -2
f) DNE
g) DNE
h) 0



Mathematica Commands Here

Soln to TEE optional Practice (con't)

3) a) (Use the chain rule with the product rule and another chain rule embedded.)

$$\frac{d}{dx}\left(e^{(\sin x)(e^{x^2})}\right) = e^{(\sin x)(e^{x^2})}\frac{d}{dx}\left(\sin(x)\, e^{x^2}\right)$$

$$= e^{(\sin x)(e^{x^2})}\left(\sin(x)\frac{d}{dx}\left(e^{x^2}\right) + e^{x^2}\left(\frac{d}{dx}(\sin(x))\right)\right)$$

f'[x]

$$= e^{(\sin x)(e^{x^2})}\left(\sin(x)\, e^{x^2}\left(\frac{d}{dx}(x^2)\right) + e^{x^2}\cos(x)\right)$$

$$= e^{(\sin x)(e^{x^2})}\left(\sin(x)\, e^{x^2}\, 2x + e^{x^2}\cos(x)\right)$$

b) (Use chain rule with another chain rule embedded.)

$$\frac{d}{dx}\left(\tan\left(e^{-x^3}\right)\right) = \sec^2\left(e^{-x^3}\right)\frac{d}{dx}\left(e^{-x^3}\right)$$

g'[x]

$$= \sec^2\left(e^{-x^3}\right)\left(e^{-x^3}\frac{d}{dx}(-x^3)\right) = \boxed{\sec^2\left(e^{-x^3}\right)e^{-x^3}\left(-3x^2\right)}$$

c) $$\frac{\partial h}{\partial x} = \frac{\partial}{\partial x}\left(3x^2y + \cos(xz^2) + xy^3z^2\right)$$

$$= 3(2x)y - \sin(xz^2)\frac{\partial}{\partial x}(xz^2) + y^3z^2$$

$$= 6xy - \sin(xz^2)(z^2) + y^3z^2$$

$$\frac{\partial}{\partial z}\left(\frac{\partial h}{\partial x}\right) = \frac{\partial}{\partial z}\left(6xy - z^2\sin(xz^2) + y^3z^2\right)$$

$$= 0 - z^2\frac{\partial}{\partial z}(\sin(xz^2)) - \frac{\partial}{\partial z}(z^2)\sin(xz^2) + y^3 2z$$

D[h[x,y,z], x, z]

$$= -z^2\cos(xz^2)\frac{\partial}{\partial z}(xz^2) - 2z\sin(xz^2) + y^3 2z$$

$$= -z^2\cos(xz^2)(x2z) - 2z\sin(xz^2) + y^3 2z$$

$$= -2xz^3\cos(xz^2) - 2z\sin(xz^2) + 2zy^3$$

4) $\frac{dg}{dt} = -0.04g(t)$

5) $\frac{dy}{dt} = -3x^2+8$, $y(0)=24$

$x_0 = 0 \quad y_0 = 24 \quad h = 1/4$

$x_1 = 1/4 \quad y_1 = y_0 + h(-3x_0^2+8)$
$= 24 + 1/4(-3(0)^2+8) = 26$

$x_2 = 1/2 \quad y_2 = y_1 + h(-3x_1^2+8)$
$= 26 + 1/4(-3(1/4)^2+8) = 27.953\ldots$

$x_3 = 3/4 \quad y_3 = y_2 + h(-3x_2^2+8)$
$= 27.953\ldots + 1/4(-3(1/2)^2+8) = 29.765\ldots$

$x_4 = 1 \quad y_4 = y_3 + h(-3x_3^2+8)$
$= 29.765\ldots + 1/4(-3(3/4)^2+8) = 31.343\ldots$

$y(1) \approx 31.34$

From Excel: $y(1) \approx 31.145$ (with stepsize 0.1)

6) a) distance $= \sqrt{(t^2-5t^2)^2+(5t-12t)^2+(t^2-8t+20-40+25t-4t^2)^2}$

minimize $f(t) = d^2 = (-4t^2)^2 + (-7t)^2 + (-3t^2+17t-20)^2$

subject to constraint $0 \le t \le 10$.

From Mathematica, $f(t)$ has critical number $t = .950376$
(from Solve[f'[t]==0,t]).

Checking $t=0$, $t=.95$, and $t=10$:
$f(0) = 400$, $f(.95) = 100.255$, $f(10) = 187400$

The smallest distance in the first 10 seconds is $\sqrt{100.25} \approx 10.01$. They do not collide.

b) The maximum height occurs when the z-coordinate has a maximum $\longrightarrow$ (con't next page)

6) b) (con't)

For p1: maximize $z(t) = t^2 - 8t + 20$
subject to $0 \le t \le 10$

From Mathematica, Solve[z'[t]==0, t] has soln $t = 4$

$z(0) = 20$, $z(4) = 4$, $z(10) = 40$.

The max height of UAV 1 is 40 units during the first 10 seconds

For p2: maximize $z(t) = 40 - 25t + 4t^2$, subject to $0 \le t \le 10$.

From Mathematica, Solve[z'[t]==0, t] has soln $t = 3.125$

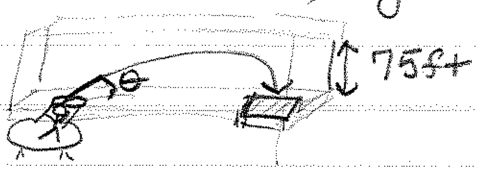
$z(0) = 40$, $z(3.125) = .9375$, $z(10) = 190$.

The max height of UAV 2 is 190 units during the first 10 seconds

7) (Assume the cannon is located under the floor so that the performer is fired from location (0,0).)

We'll neglect air resistance and use the equations

$x = v_0 \cos\theta \, t$, $y = v_0 \sin\theta \, t - \frac{1}{2} g t^2$ to describe the motion.



To find the maximum height, find when $y'(t) = 0$.

$y'(t) = v_0 \sin\theta - \frac{1}{2} \cdot 2gt = 0 \Rightarrow t = \dfrac{v_0 \sin\theta}{g}$

Max height is $y\left(\dfrac{v_0 \sin\theta}{g}\right) = v_0 \sin\theta \left(\dfrac{v_0 \sin\theta}{g}\right) - \frac{1}{2} g \left(\dfrac{v_0 \sin\theta}{g}\right)^2$

$= \dfrac{v_0^2 \sin^2\theta}{g} - \dfrac{v_0^2 \sin^2\theta}{2g} = \dfrac{v_0^2 \sin^2\theta}{2g}$

(Still need θ before can determine max height.)

Can performer make it to cushion? (When does $x = 200$?)

Find when $y = 0$: $0 = v_0 \sin\theta \, t - \frac{1}{2} g t^2$

$0 = t(v_0 \sin\theta - \frac{1}{2} g t)$

$\Rightarrow t = 0$ or $t = \dfrac{2 v_0 \sin\theta}{g}$

→ con't

7) (con't) So when $t = \frac{2v_0 \sin\theta}{g}$ the performer should reach cushion, i.e. $x\left(\frac{2v_0 \sin\theta}{g}\right) = 200$ should be true.

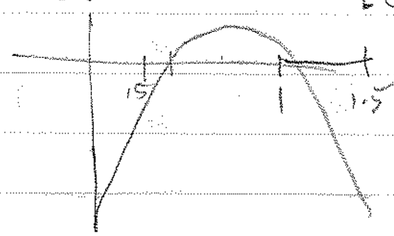
Find θ that makes this true:

$$x\left(\frac{2v_0\sin\theta}{g}\right) = 200 = v_0\cos\theta\left(\frac{2v_0\sin\theta}{g}\right)$$

$$200 = \frac{2v_0^2}{g}\cos\theta\sin\theta$$

Graphing $\frac{2\left(\frac{80}{3}\sqrt{10}\right)^2}{32}\cos\theta\sin\theta - 200$ in Mathematica I

see 2 roots: $\text{Plot}[\underbrace{2*(80/3*\text{Sqrt}[10])\text{^}2*\text{Cos}[\text{theta}]*\text{Sin}[\text{theta}]/32-200}_{f[\text{theta}]}, \{\text{theta}, 0, \text{Pi}/2\}]$



FindRoot[f[theta], {theta, .5}]

FindRoot[f[theta], {theta, 1}]

The performer will reach cushion if
$\theta = .559885$ or $\theta = 1.01091\ldots$
<u>if</u> he doesn't hit the ceiling.

Max height for $\theta = .559885$ is $\frac{v_0^2\sin^2\theta}{2g} = 31.34$ ft

Max height for $\theta = 1.01091$ is 79.77 ft (too high!)

$\boxed{\text{The performer can be fired to the cushion without striking the ceiling as long as the cannon's angle of elevation is .559885 radians } (\approx 32^\circ)}$

Soln to
TEE Optional
Practice
(con't)

8) If the plane is perpendicular to the tangent to the curve, the tangent vector at $(1,1,2/3)$ can be used as the normal vector for the plane.

$r(t) = \langle t, t, \tfrac{2}{3} t^{3/2} \rangle$

Notice that point $(1,1,2/3)$ is $r(1)$, i.e. $t=1$

$r'(t) = \langle 1, 1, \tfrac{2}{3}(\tfrac{3}{2}) t^{3/2 - 2/2} \rangle = \langle 1, 1, t^{1/2} \rangle$

at $t=1$, the tangent vector is $\langle 1,1,1 \rangle$.

Vector Equation of the Plane: $\langle 1,1,1\rangle \cdot (\langle x,y,z\rangle - \langle 1,1,2/3\rangle) = 0$

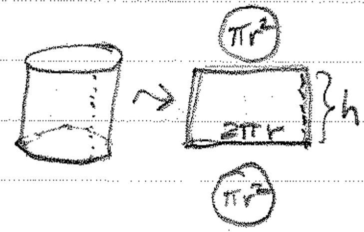
(Scalar Equation: $1(x-1) + 1(y-1) + 1(z - 2/3) = 0$)

[You could also do this with Lagrange Multipliers]

9) minimize $SA = \pi r^2 + \pi r^2 + 2\pi r h$

subject to $\pi r^2 h = 16\pi$

(with $r > 0$ and $h > 0$)



- Solve constraint for h: $h = 16/r^2$
- Substitute into SA to get equation of 1 variable:

$f(r) = SA(r, 16/r^2) = 2\pi r^2 + 2\pi r(16/r^2) = 2\pi r^2 + 32\pi/r$

Solve[f'[r]==0, r] $\Rightarrow r = 2$

f''[2] $= 12\pi > 0$ so by 2nd derivative test, this is a minimum.

The dimensions are radius = 2cm, height = 4cm.

10) minimize $M(x,y,z) = 6x - y^2 + xz + 60$

subject to $x^2 + y^2 + z^2 = 6^2$

Let $g(x,y,z) = x^2 + y^2 + z^2$.

Use Lagrange Multipliers and Mathematica to solve:

$$\begin{cases} \nabla f(x,y,z) = \lambda \nabla g(x,y,z) \\ g(x,y,z) = 36 \end{cases}$$

$\Longrightarrow$ next page

10 (con't)

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M[x_,y_,z_]= 6x-y^2+x*z+60
g[x_,y_,z_]= x^2+y^2+z^2

gradM = {D[M[x,y,z],x], D[M[x,y,z],y], D[M[x,y,z],z]}
grad g= {D[g[x,y,z],x], D[g[x,y,z],y], D[g[x,y,z]]}

Solve[{gradM == L* gradg, g[x,y,z]==36}, {x,y,z,L}]
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This returns the following solutions:

$(0,0,-6)$
$(-3\sqrt{3},0,3)$
$(3\sqrt{3},0,3)$
$(-4,-4,2)$
$(-4,4,2)$

Check to see which gives minimum $M(x,y,z)$:

$M(0,0,-6)= 60$
$M(-3\sqrt{3},0,3)= 13.2346$
$M(3\sqrt{3},0,3)= 106.765$
$M(-4,-4,2)= 12$
$M(-4,4,2)= 12$

The radio telescope should be located at either point $(-4,-4,2)$ or point $(-4,4,2)$.

You could also solve this problem by solving $x^2+y^2+z^2=6^2$ for y^2 & substitute that into $M(x,y,z)$ to make a function of 2 variables. Then, you just need to minimize that function using methods from Section 11.7 and look at the graph to be sure you have the absolute minimum and not a relative minimum. I think Lagrange is more nifty though...