B3, C3, D3 MA104 SAMPLE SOLUTIONS

Optional Practice - MA104 Course Quiz II

1) Let $x = e^t$ then $x^2 = (e^t)^2 = e^t e^t = e^{t+t} = e^{2t}$

Cartesian equation: $y = x^2 + 1$

2) Let $t = x - 1$, then $y = 4t^3$ and $t + 1 = x$

Parametric equation: $x(t) = t+1,\ y(t) = 4t^3$

[There is more than one possible answer.
For example, $x(t) = t$, $y(t) = 4(t-1)^3$ is also valid.]

3) $p_1(t) = \{x_1(t), y_1(t)\} = \{2t+1, 4t^2\}$
$p_2(t) = \{x_2(t), y_2(t)\} = \{3t, 3t\}$

The paths intersect if there is some s and r so that $p_1(s) = p_2(r)$:

$$\begin{cases} 2s+1 = 3r \\ 4s^2 = 3r \end{cases}$$

$\Rightarrow 4s^2 = 2s+1$

$4s^2 - 2s - 1 = 0$

$$s = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(4)(-1)}}{2(4)} = \frac{2 \pm \sqrt{4+16}}{8} = \frac{2 \pm \sqrt{20}}{8} = \frac{2 \pm 2\sqrt{5}}{8}$$

$\Rightarrow s_1 = \frac{1}{4} + \frac{1}{4}\sqrt{5}$ and $s_2 = \frac{1}{4} - \frac{1}{4}\sqrt{5}$

and $r = \frac{2}{3}s + \frac{1}{3}$

$r_1 = \frac{2}{3}\left(\frac{1}{4} + \frac{1}{4}\sqrt{5}\right) + \frac{1}{3}$

$= \frac{1}{2} + \frac{1}{6}\sqrt{5}$

[Note: You'll get the same r-values if you use the second eqn to get $r = \frac{4}{3}s^2$ instead of the first equation.]

$\Rightarrow r_1 = \frac{1}{2} + \frac{1}{6}\sqrt{5}$ and $r_2 = \frac{1}{2} - \frac{1}{6}\sqrt{5}$

The particles paths do intersect twice since $p_1\left(\frac{1}{4}+\frac{1}{4}\sqrt{5}\right) = p_2\left(\frac{1}{2}+\frac{1}{6}\sqrt{5}\right)$ and $p_1\left(\frac{1}{4}-\frac{1}{4}\sqrt{5}\right) = p_2\left(\frac{1}{2}-\frac{1}{6}\sqrt{5}\right)$. Since the particles do not go through the same point at the same time, they do not collide.

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4) The shortest distance between (6,5,7) and the xy-plane is 7.
" " " " " " " xz-plane is 5.
" " " " " " " yz-plane is 6.
$\Rightarrow$ The largest such sphere could only have radius 5.

$$(x-6)^2+(y-5)^2+(z-7)^2=5^2$$

5) $\|\vec{PQ}\| = \sqrt{(7-3)^2+(0-(-2))^2+(1-(-3))^2} = \sqrt{4^2+2^2+4^2} = 6$

$\|\vec{PR}\| = \sqrt{(1-3)^2+(2-(-2))^2+(1-(-3))^2} = \sqrt{(-2)^2+4^2+4^2} = 6$

$\|\vec{QR}\| = \sqrt{(1-7)^2+(2-0)^2+(1-1)^2} = \sqrt{(-6)^2+2^2+0^2} = \sqrt{40} = 2\sqrt{10}$

6) SEE END of Sample Solutions

7) The position of the baseball is
$\underset{\sim}{r}(t) = \langle -10.2t^2-91.5t+40.6, -9.6t^2-91.6t+40.6, -16.1t^2+1.7t+6.5\rangle$.
The velocity is:
$\underset{\sim}{v}(t) = \underset{\sim}{r}'(t) = \langle -20.4t-91.5, -19.2t-91.6, -32.2t+1.7\rangle$.
The velocity at time $t=0.32s$ is:
$\underset{\sim}{v}(0.32) = \langle -98.028, -97.744, -8.604\rangle$
The speed at time $t=0.32s$ is:
$\|\underset{\sim}{v}(0.32)\| = \sqrt{(-98.028)^2+(-97.744)^2+(-8.604)^2}$
$= \sqrt{19237.407} \cong 138.7 \text{ ft/s}$

8) a) The missle will hit the target if there exists a time t so that $\underset{\sim}{a}(t) = \underset{\sim}{m}(t)$.

$\Longrightarrow$ (con't)

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8) (con't)

$$\begin{cases} 900+150t = 250t \\ -450+300t = 250t \\ 3690-50t = 40t^2 \end{cases}$$

$$\begin{cases} 900 = 100t \\ 50t = 450 \\ 0 = 40t^2+50t-3690 \end{cases}$$

$$\begin{cases} 9 = t \\ t = 9 \\ t = \dfrac{-50 \pm \sqrt{50^2-4(40)(-3690)}}{80} \end{cases} \Rightarrow t = 9 \text{ or } t = -10.25$$

The missile hits the target at $t = 9$ s.

b) $\underset{\sim}{a}(9) = \langle 900+150(9), -450+300(9), 3690-50(9)\rangle$
$= \langle 2250, 2250, 3240 \rangle$

The height of the explosion is the y-component of $\underset{\sim}{a}(9)$.

i.e. 2250 m

[Check: This height should be equal to the y-component of $\underset{\sim}{m}(9)$... $250(9) = 2250$ ✓ Good.]

9) The scalar projection of $\underset{\sim}{b}$ onto $\underset{\sim}{a}$ is "$\text{comp}_{\underset{\sim}{a}}\underset{\sim}{b}$":

$\text{comp}_{\underset{\sim}{a}}\underset{\sim}{b} = \dfrac{\underset{\sim}{a}}{\|\underset{\sim}{a}\|} \cdot \underset{\sim}{b}$ $\quad \|\underset{\sim}{a}\| = \sqrt{2^2+(-3)^2+1^2} = \sqrt{4+9+1} = \sqrt{14}$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = (1)(2)+(6)(-3)+(-2)(1) = -17$

$\text{comp}_{\underset{\sim}{a}}\underset{\sim}{b} = \dfrac{-17}{\sqrt{14}}$

The vector projection of $\underset{\sim}{b}$ onto $\underset{\sim}{a}$ is $\text{proj}_{\underset{\sim}{a}}\underset{\sim}{b} = \dfrac{-17}{\sqrt{14}} \left\langle \dfrac{2}{\sqrt{14}}, \dfrac{-3}{\sqrt{14}}, \dfrac{1}{\sqrt{14}} \right\rangle$

$= \langle -17/7, 51/14, -17/14 \rangle$

10) We're seeking a normal vector to the plane.
First, find two vectors in the plane, for example
$\underset{\sim}{a} = \overrightarrow{PQ} = \langle 0-1, 2-0, 0-0 \rangle = \langle -1, 2, 0 \rangle$
$\underset{\sim}{b} = \overrightarrow{PR} = \langle 0-1, 0-0, 3-0 \rangle = \langle -1, 0, 3 \rangle$
Now, the cross product of the 2 vectors will be orthogonal to the plane containing the vectors:
$\underline{\underline{\underset{\sim}{a} \times \underset{\sim}{b}}}$: In Mathematica: a={-1, 2, 0}
b={-1, 0, 3}
Cross[a, b] [Output is {6, 3, 2}]

By Hand:
$$\left.\begin{array}{cccccc} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} & \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -1 & 2 & 0 & -1 & 2 & 0 \\ -1 & 0 & 3 & -1 & 0 & 3 \end{array}\right\} \begin{array}{l} 6\underset{\sim}{i} + 0\underset{\sim}{j} + 0\underset{\sim}{k} \\ -0\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k} \\ \hline 6\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k} \end{array}$$

$\langle 6, 3, 2 \rangle$

11) SEE END of sample solutions

12) The direction of $\underset{\sim}{a} = \langle 5, -6, \sqrt{39} \rangle$ is $\dfrac{\underset{\sim}{a}}{\|\underset{\sim}{a}\|}$.

$\|\underset{\sim}{a}\| = \sqrt{5^2 + (-6)^2 + (\sqrt{39})^2} = \sqrt{25+36+39} = \sqrt{100} = 10$

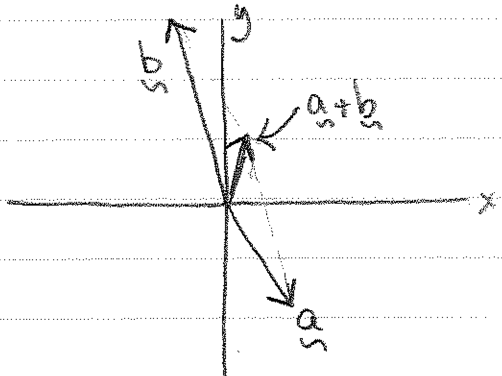
So, the direction of $\underset{\sim}{a}$ is $\dfrac{\langle 5, -6, \sqrt{39} \rangle}{10} = \left\langle \dfrac{5}{10}, \dfrac{-6}{10}, \dfrac{\sqrt{39}}{10} \right\rangle$,

i.e. $\dfrac{\underset{\sim}{a}}{\|\underset{\sim}{a}\|} = \left\langle \dfrac{1}{2}, \dfrac{-3}{5}, \dfrac{\sqrt{39}}{10} \right\rangle$. [Note: $\dfrac{\underset{\sim}{a}}{\|\underset{\sim}{a}\|}$ has length $\underline{1}$]

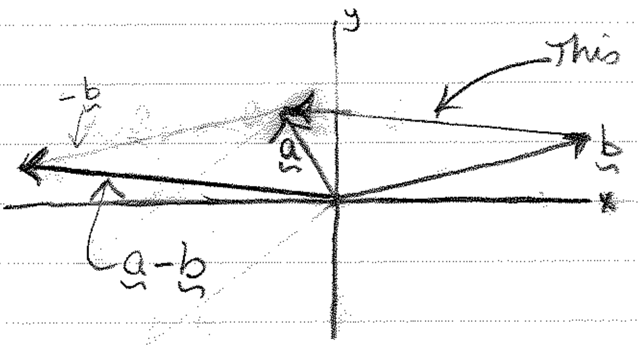
The vector we seek is $\left\langle \dfrac{\sqrt{3}}{2}, \dfrac{-3\sqrt{3}}{5}, \dfrac{3\sqrt{13}}{10} \right\rangle$

[Note: $\sqrt{3}\sqrt{39} = \sqrt{3 \cdot 39} = \sqrt{3 \cdot 3 \cdot 13} = 3\sqrt{13}$.]

13) [This is a free form problem designed to get you to think about the geometric implication of adding vectors. See p. 643]



14) [See p.644... Realize also that $\underset{\sim}{a} - \underset{\sim}{b}$ is the same as $\underset{\sim}{a} + (-\underset{\sim}{b})$.]



15) Find $\underset{\sim}{v} = \langle v_1, v_2 \rangle$ so that $\|\underset{\sim}{v}\| = 2$ and $\underset{\sim}{v} \cdot \langle 1, 3 \rangle = 0$

$$\begin{cases} v_1 + 3v_2 = 0 \\ \sqrt{v_1^2 + v_2^2} = 2 \end{cases}$$

Solve the first eqn + plug into the second:

$$v_1 = -3v_2 \Rightarrow \sqrt{(-3v_2)^2 + v_2^2} = 2$$

$$\sqrt{9v_2^2 + v_2^2} = 2$$

$$10v_2^2 = 4$$

$$v_2^2 = \frac{4}{10}$$

$$\Rightarrow v_2 = \pm \frac{2}{\sqrt{10}}$$

Now, $v_1 = -3\left(\frac{2}{\sqrt{10}}\right)$ or $v_1 = -3\left(\frac{-2}{\sqrt{10}}\right)$

There are 2 possible vectors: $\left\langle \frac{-6}{\sqrt{10}}, \frac{2}{\sqrt{10}} \right\rangle$ OR $\left\langle \frac{6}{\sqrt{10}}, \frac{-2}{\sqrt{10}} \right\rangle$

16) $\text{proj}_{\mathbf{b}}\,\mathbf{a} = \left(\frac{\mathbf{b}}{\|\mathbf{b}\|} \cdot \mathbf{a}\right)\frac{\mathbf{b}}{\|\mathbf{b}\|}$

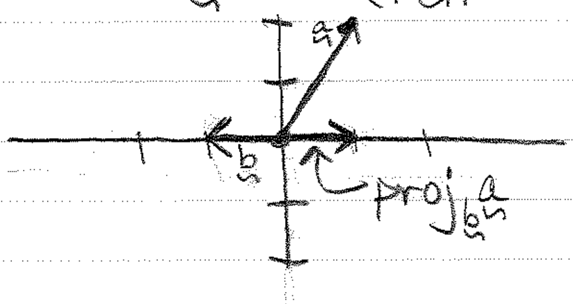
$\|\mathbf{b}\| = \sqrt{(-1)^2+0^2} = 1$, So $\frac{\mathbf{b}}{\|\mathbf{b}\|} = \langle -1, 0\rangle$

$\text{proj}_{\mathbf{b}}\,\mathbf{a} = \left(\langle -1,0\rangle \cdot \langle 1,2\rangle\right)\langle -1,0\rangle = \left((-1)(1)+(2)(0)\right)\langle -1,0\rangle$

$= (-1)\langle -1,0\rangle = \langle 1,0\rangle$

$\text{proj}_{\mathbf{b}}\,\mathbf{a} = \langle 1,0\rangle$

$\text{comp}_{\mathbf{b}}\,\mathbf{a} = \left(\frac{\mathbf{b}}{\|\mathbf{b}\|} \cdot \mathbf{a}\right) = -1$



Ah-ha! $\text{comp}_{\mathbf{b}}\,\mathbf{a}$ can be negative. It is not the length of $\text{proj}_{\mathbf{b}}\,\mathbf{a}$, since that would be $|-1|$, i.e. $+1$.

17) $(\mathbf{a}\times\mathbf{b})\cdot \mathbf{a}$ is always 0. $\mathbf{a}\times\mathbf{b}$ results in a vector which is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$. Therefore, the dot product of this vector with $\mathbf{a}$ will certainly be zero since the dot product of 2 vectors is always zero if they are orthogonal. $\mathbf{a}\times\mathbf{b}\cdot\mathbf{b}$ is also zero for the same reason.

18) $\mathbf{a}\cdot\mathbf{b} = (5)(0)+(-1)(1)+(1)(-5) \neq 0$

$\mathbf{a}\cdot\mathbf{c} = (5)(-15)+(-1)(3)+(1)(-3) \neq 0$

$\mathbf{b}\cdot\mathbf{c} = (0)(-15)+(1)(3)+(-5)(-3) \neq 0$

no pairs are orthogonal

$\mathbf{a}\times\mathbf{b} = (5-1)\mathbf{i}+(0+25)\mathbf{j}+(5-0)\mathbf{k}$

$\neq \mathbf{0}$

$\mathbf{a}\times\mathbf{c} = (3-3)\mathbf{i}+(-15+15)\mathbf{j}+(15-15)\mathbf{k}$

$= 0\mathbf{i}+0\mathbf{j}+0\mathbf{k}$

$\mathbf{b}\times\mathbf{c} = (-3+15)\mathbf{i}+(75-0)\mathbf{j}+(0+15)\mathbf{k}$

$\neq \mathbf{0}$

$\mathbf{a}$ and $\mathbf{c}$ are parallel

Scratch work:

$\begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & -1 & 1 \\ 0 & 1 & -5 \end{matrix}$

$\begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & -1 & 1 \\ -15 & 3 & -3 \end{matrix}$

$\begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & -5 \\ -15 & 3 & -3 \end{matrix}$

19) $\underset{\sim}{a} = 4\underset{\sim}{i}$
$\underset{\sim}{b} = 6\underset{\sim}{k}$

a) The direction of $\underset{\sim}{a} \times \underset{\sim}{b}$ is $-\underset{\sim}{j}$
The direction of $\underset{\sim}{b} \times \underset{\sim}{a}$ is $\underset{\sim}{j}$

b) The magnitude of $\underset{\sim}{a} \times \underset{\sim}{b}$ is 24.
" " " $\underset{\sim}{b} \times \underset{\sim}{a}$ " also 24.

(Think: $\|\underset{\sim}{a} \times \underset{\sim}{b}\| = \|(\|\underset{\sim}{a}\|\|\underset{\sim}{b}\| \sin\theta_{a,b})\underset{\sim}{n}\| = \underbrace{\|\underset{\sim}{a}\|}_{4}\underbrace{\|\underset{\sim}{b}\|}_{6}\underbrace{\sin 90°}_{1}\cdot\underbrace{\|\underset{\sim}{n}\|}_{1}$

OR
$$\begin{matrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} & \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 4 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 6 \end{matrix}$$
$$\begin{aligned} &0\underset{\sim}{i} + 0\underset{\sim}{j} + 0\underset{\sim}{k} \\ +\,&0\underset{\sim}{i} - 24\underset{\sim}{j} + 0\underset{\sim}{k} \\ \hline &0\underset{\sim}{i} - 24\underset{\sim}{j} + 0\underset{\sim}{k} \end{aligned}$$
)

20)* [This should say "parametric equation of a line".]

$$\underset{\sim}{r}(t) = \langle -5, 2, 1\rangle + t\langle 1, 1, 6\rangle$$

$$x = -5 + t, \quad y = 2 + t, \quad z = 1 + 6t$$

21) The path of the projectile at any time is
$r'(t) = \langle 5, 4 + 2(\tfrac{1}{2})9.8t\rangle = \langle 5, 4 + 9.8t\rangle$.
The path at $t = 0.5$ is $r'(0.5) = \langle 5, 4 + 9.8(0.5)\rangle = \langle 5, 8.9\rangle$
The ground direction is $\langle 1, 0\rangle$.
$\langle 5, 8.9\rangle \cdot \langle 1, 0\rangle = \|\langle 5, 8.9\rangle\| \, \|\langle 1, 0\rangle\| \cos\theta_{\text{ground}, r'(t)}$
$(5)(1) + (8.9)(0) = \sqrt{5^2 + 8.9^2}\sqrt{1^2 + 0^2} \cos\theta_{g, r'}$
$5 = \sqrt{104.21} \cos\theta_{g, r'(t)}$
$\theta_{g, r'(t)} = \cos^{-1}\left(\frac{5}{\sqrt{104.21}}\right) \approx 60.7° \approx 1.06$ radians

6) a) $P(-10,10,3)$ $Q(10,-10,3)$ $R(10,10,2.5)$

Plan: Find two vectors in the plane.
Find the cross product which gives a normal to the plane.
Use one point + the normal to write an equation.

$\underset{\sim}{a} = \overrightarrow{PQ} = \langle 10-(-10), -10-10, 3-3 \rangle = \langle 20, -20, 0 \rangle$

$\underset{\sim}{b} = \overrightarrow{PR} = \langle 10-(-10)\ 10-10, 2.5-3 \rangle = \langle 20, 0, -0.5 \rangle$

$$\left.\begin{array}{cccccc} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} & \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 20 & -20 & 0 & 20 & -20 & 0 \\ 20 & 0 & -1/2 & 20 & 0 & -1/2 \end{array}\right\} \quad \underset{\sim}{a} \times \underset{\sim}{b} = (10-0)\underset{\sim}{i} + (0+10)\underset{\sim}{j} + (0+400)\underset{\sim}{k}$$

A normal vector is $\underset{\sim}{a} \times \underset{\sim}{b} = \langle 10, 10, 400 \rangle$

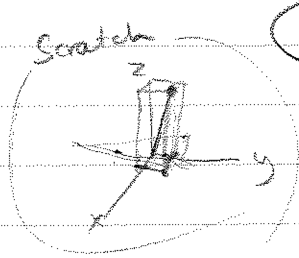
[This problem should say "an equation"]

An equation is $\langle 10,10,400 \rangle \cdot (\underset{\sim}{r} - \langle -10,10,3 \rangle) = 0$

b) The shortest path between the laser and the plane is the line that goes from point $(1,-1,0)$ and intersects the plane at a right angle. In other words, the line we want goes through $(1,-1,0)$ + has direction $\langle 10,10,400 \rangle$.

$\underset{\sim}{r}(t) = \langle 1,-1,0 \rangle + t\langle 10,10,400 \rangle$

Scratch



The troopers should fire the laser in the direction $\langle 10, 10, 400 \rangle$.

Find where the line intersects the plane. i.e. solve
$\langle 10,10,400 \rangle \cdot (\underset{\sim}{r} - \langle -10,10,3 \rangle) = 0$ for t where $\underset{\sim}{r} = \langle 1+10t, -1+10t, 400t \rangle$.

$10(1+10t+10) + 10(-1+10t-10) + 400(400t-3) = 0$

$\Rightarrow$ (skipping algebra) $t = 6/801$. So the point to observe

is $x = 1 + 10(6/801)$, $y = -1 + 10(6/801)$, $z = 400(6/801)$ $\Rightarrow \left(\frac{861}{801}, \frac{-741}{801}, \frac{2400}{801}\right)$

6) (con't)

c) The force field intersects the xy-plane when the z-coordinate is 0. The intersection line must be perpendicular to both the normal vector of the force field plane and the normal vector of the xy-plane. In other words, the intersection line must be parallel to the cross product of the two normal vectors.

$$\left.\begin{array}{cccccc} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 10 & 10 & 400 & 10 & 10 & 400 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right\} \Rightarrow (10-0)\mathbf{i} + (0-10)\mathbf{j} + (0-0)\mathbf{k}$$

a normal to the xy plane is $\langle 0,0,1\rangle$

The line we seek has direction $\langle 10,-10,0\rangle$.

We just need to find a point in the xy-plane and on the force field, i.e. a point on the force field with its z-coordinate equal to 0.

$$10(x+10) + 10(y+10) + 400(0-3) = 0$$

$$10x + 100 + 10y + 100 - 1200 = 0$$

$$\Rightarrow x + y - 100 = 0$$

Let $x=0$, then $y=100$. So $(0,100,0)$ is on the xy plane + the force field.

$$\mathbf{r}(t) = \langle 0,100,0\rangle + t\langle 10,-10,0\rangle$$

$$x = 10t, \quad y = 100-10t, \quad z=0$$

[Remember, there many correct answers to this problem.]

11) $x(t) = 45t \quad y(t) = 50t - 16t^2 + 6$ } Target
$u(t) = 100 - 40t \quad v(t) = 60t - 16t^2 + 5$ } Projectile

a) Is there a t so that $T(t) = U(t)$?

$$\begin{cases} 45t = 100 - 40t \\ 50t - 16t^2 + 6 = 60t - 16t^2 + 5 \end{cases}$$

$$\begin{cases} 85t = 100 \\ 1 = 10t \end{cases}$$

$$\begin{cases} t = 100/85 \approx 11.76 \\ t = 1/10 = 0.1 \end{cases}$$

They do not collide.

b) Is there a time t and s so that $T(t) = U(s)$?

$$\begin{cases} 45t = 100 - 40s \\ 50t - 16t^2 + 6 = 60s - 16s^2 + 5 \end{cases}$$

1st eqn $\Rightarrow t = 100/45 - 40/45\, s = 20/9 - 8/9\, s$; subst into 2nd eqn:

$$50(20/9 - 8/9\, s) - 16(\tfrac{20}{9} - \tfrac{8}{9}s)^2 + 6 = 60s - 16s^2 + 5$$

$$1000/9 - 400/9\, s - 16(\tfrac{400}{81} - \tfrac{320}{81}s + \tfrac{64}{81}s^2) + 6 = 60s - 16s^2 + 5$$

(skip steps)

$$272s^2 - 3340s + 2681 = 0$$

$$s = \frac{-(-3340) \pm \sqrt{(-3340)^2 - 4(272)(2681)}}{2(272)}$$

$s_1 \approx 11.42 \quad s_2 \approx 0.863$

$\Rightarrow t_1 \approx -7.93 \quad t_2 \approx 1.45$

Ignore (negative time)

The projectile trajectory does intercept the target's projectile. The projectile arrives first.

11) (con't)

c) The distance between the Target + Projectile is

$\sqrt{(x(t)-u(t))^2+(y(t)-v(t))^2}$. If there is a time so that distance is $=12$, then the projectile can destroy the target (with 95% probability) if the projectile is detonated at that t.

$$\sqrt{(45t-(100-40t))^2+(50t-16t^2+6-(60t-16t^2+5))^2}=12$$

$$(45t-100+40t)^2+(50t-16t^2+6-60t+16t^2-5)^2=12^2$$

$$(85t-100)^2+(1-10t)^2=144$$

$$9025t^2-19000t+10000+1-20t+100t^2=144$$

$$9125t^2-19020t+9857=0$$

$$t=\frac{19020\pm\sqrt{(-19020)^2-4(9125)(9857)}}{2(9125)}$$

$t_1 \simeq 1.12 \quad t_2 \simeq 0.97$

In fact, this equation is less than zero between $t=0.97$ and 1.12.

Detonate the projectile between 0.97 units (seconds) and 1.12 units (seconds).
To have the best chance of destroying the target, you could MINIMIZE the distance between the Target and the Projectile and detonate the projectile at that specific time.