4/12/06
Sample solutions

Problem Set 8 - Sample Solutions

1) (In general, yes, you can find ∇f at a point Q if the directional derivative is known in 2 non-parallel directions...)

$$\begin{cases} D_{\underset{\sim}{u}_1} f(3,4) = 23/10 \\ D_{\underset{\sim}{u}_2} f(3,4) = 7/\sqrt{5} \end{cases}$$

$$\begin{cases} \nabla f(3,4) \cdot \langle 4/5, -3/5 \rangle = 23/10 \\ \nabla f(3,4) \cdot \langle 1/\sqrt{5}, 2/\sqrt{5} \rangle = 7/\sqrt{5} \end{cases}$$

$$\begin{cases} \langle f_x(3,4), f_y(3,4) \rangle \cdot \langle 4/5, -3/5 \rangle = 23/10 \\ \langle f_x(3,4), f_y(3,4) \rangle \cdot \langle 1/\sqrt{5}, 2/\sqrt{5} \rangle = 7/\sqrt{5} \end{cases}$$

$$\begin{cases} f_x(3,4)\,(4/5) + f_y(3,4)(-3/5) = 23/10 & (1) \\ f_x(3,4)\,(1/\sqrt{5}) + f_y(3,4)(2/\sqrt{5}) = 7/\sqrt{5} & (2) \end{cases}$$

$$(1) \Rightarrow f_x(3,4) = \frac{\frac{23}{10} + \frac{3}{5} f_y(3,4)}{4/5} = \frac{23}{8} + \frac{3}{4} f_y(3,4)$$

$$(2) \Rightarrow \left(\frac{23}{8} + \frac{3}{4} f_y(3,4)\right)\frac{1}{\sqrt{5}} + \frac{2}{\sqrt{5}} f_y(3,4) = 7/\sqrt{5}$$

$$\frac{23}{8} + \frac{3}{4} f_y(3,4) + 2 f_y(3,4) = 7$$

$$\frac{11}{4} f_y(3,4) = 7 - \frac{23}{8} = \frac{56-23}{8} = \frac{33}{8}$$

$$f_y(3,4) = \frac{33}{8} \cdot \frac{4}{11} = \frac{3}{2}$$

Substituting in for $f_x(3,4)$: $f_x(3,4) = \frac{23}{8} + \frac{3}{4}\left(\frac{3}{2}\right) = \frac{23}{8} + \frac{9}{8} = \frac{32}{8} = 4$

$$\nabla f(3,4) = \langle 4, 3/2 \rangle$$

Problem Set 8 - Sample Solutions

2) a) The gradient is normal to the level surface.

$\left(\begin{array}{l} f(-1,2,1) = (-1)^2 + 2(2)^2 + 1^2 - (2)(1) = 1 + 8 + 1 - 2 = 8 \\ Q \text{ is on the level surface.} \end{array} \right)$

$$\nabla f(x,y,z) = \langle 2x, 4y - z, 2z - y \rangle$$

$$\nabla f(-1,2,1) = \langle -2, 8-1, 2-2 \rangle = \langle -2, 7, 0 \rangle$$

$\boxed{\text{The vector } \langle -2,7,0 \rangle \text{ is normal to the level surface } f(x,y,z) = 8 \text{ at point } Q(-1,2,1).}$

b) The gradient also defines a plane which is tangent to the level surface at point Q. The equation of the plane is

$$\langle -2,7,0 \rangle \cdot \left(\langle x,y,z \rangle - \langle -1,2,1 \rangle \right) = 0$$

3) a) $\|\nabla f(1,4)\|$ is the maximum rate of change.

$$\nabla f(x,y) = \left\langle \frac{(2y+x)(1) - x(1)}{(2y+x)^2}, \frac{(2y+x)(0) - x(2)}{(2y+x)^2} \right\rangle$$

$$= \left\langle \frac{2y}{(2y+x)^2}, \frac{-2x}{(2y+x)^2} \right\rangle$$

$$\nabla f(1,4) = \left\langle \frac{8}{(9)^2}, \frac{-2}{(9)^2} \right\rangle = \left\langle \frac{8}{81}, \frac{-2}{81} \right\rangle$$

$$\|\nabla f(1,4)\| = \sqrt{\frac{64}{(81)^2} + \frac{4}{(81)^2}} = \sqrt{\frac{68}{(81)^2}} = \frac{2\sqrt{17}}{81}$$

b) $f(x,y)$ decreases most rapidly in the direction opposite the gradient, i.e. towards $\left\langle -\frac{8}{81}, \frac{2}{81} \right\rangle$.

Problem Set 8 – Sample Solution (con't)

4) maximize $P = 2pq + 2pr + 2rq$ subject to CONSTRAINT $p + q + r = 1$.

$p = 1 - q - r$

$$P = f(q,r) = 2(1-q-r)q + 2(1-q-r)r + 2rq$$
$$= 2q - 2q^2 - 2qr + 2r - 2qr - 2r^2 + 2rq$$
$$= 2q - 2q^2 - 2qr + 2r - 2r^2$$

Find critical points:

$f_q = 2 - 4q - 2r = 0$ ①

$f_r = -2q + 2 - 4r = 0$ ②

From ①, $\frac{2-4q}{2} = r \Rightarrow r = 1 - 2q$

Plug r into 2, $-2q + 2 - 4(1-2q) = 0$

$-2q + 2 - 4 + 8q = 0$

$6q = +2$

$q = 1/3$

Then $r = 1 - 2(1/3) = \frac{3}{3} - \frac{2}{3} = \frac{1}{3}$

CRITICAL POINT $(1/3, 1/3, 1 - 1/3 - 1/3)$

Is this a max or min? Check discriminant,

$$\left.\begin{array}{l} f_{qq} = -4 \\ f_{rr} = -4 \\ f_{qr} = -2 \end{array}\right\} D = (-4)(-4) - (-2)^2 = 16 - 4 = 12 > 0$$

$f_{qq} < 0$, so the critical point is a local max.

$$P(1/3, 1/3, 1/3) = 2(1/3)(1/3) + 2(1/3)(1/3) + 2(1/3)(1/3) = 2/3.$$

The maximum of P is 2/3.

(Just FYI!:)

Same problem again,

4) Using Lagrange Multipliers:

$f(p,q,r) = 2pq + 2pr + 2rq$

$g(p,q,r) = p+q+r \quad , \quad g(p,q,r) = 1$

a) $\begin{cases} \nabla f = \lambda \nabla g \\ g(p,q,r) \stackrel{?}{=} 1 \end{cases}$

$\begin{cases} \langle 2q+2r,\ 2p+2r,\ 2p+2q \rangle = \lambda \langle 1, 1, 1 \rangle \\ p+q+r = 1 \end{cases}$

$\begin{cases} 2q+2r = 1\lambda & (1) \\ 2p+2r = 1\lambda & (2) \\ 2p+2q = 1\lambda & (3) \\ p+q+r = 1 & (4) \end{cases}$

$(1) = (2) \Rightarrow 2q+2r = 2p+2r$

$2q = 2p$

$q = p$

$(3) \Rightarrow 2p+2p = \lambda$

$p = \lambda/4$

$(4) \Rightarrow \lambda/4 + \lambda/4 + \lambda/4 = 1$

$3\lambda = 4$

$\lambda = 4/3$

$\Rightarrow p = 1/3$

$\Rightarrow q = 1/3$

$\Rightarrow r = 1 - 1/3 - 1/3 = 1/3$

b) $P(1/3, 1/3, 1/3) = 2(1/3)(1/3) + 2(1/3)(1/3) + 2(1/3)(1/3) = 2/3$