Sample Solution
1/27/06
MA104
(40 pts max)

Problem Set 2

1) Consider function y with rate of change $y' = 2xy^2$ and initial condition $y(0)=1$.

Approximate $y(0.4)$ using Euler's method with stepsize $h=0.1$, $h=0.05$, $h=0.02$, $h=0.01$.

Stepsize $h=0.1$

$x_0 = 0$, $y_0 = 1$

$$\begin{cases} x_1 = 0.1 \\ y_1 = y_0 + h(2x_0 y_0^2) = 1 + (0.1)(2(0)(1)^2) = \underline{\underline{1}} \end{cases}$$

$$\begin{cases} x_2 = 0.2 \\ y_2 = y_1 + h(2x_1 y_1^2) = 1 + (0.1)(2(0.1)(1)^2) = 1 + 0.02 = \underline{\underline{1.02}} \end{cases}$$

$$\begin{cases} x_3 = 0.3 \\ y_3 = y_2 + h(2x_2 y_2^2) = 1.02 + (0.1)(2(0.2)(1.02)^2) = 1.02 + .041616 \end{cases}$$

$$= \underline{\underline{1.061616}}$$

From my Excel worksheet: $y(0.4) \approx y_4 = \underline{\underline{1.1292377\ldots}}$

Stepsize $h=0.05$:

$x_0 = 0$, $y_0 = 1$

$x_1 = 0.05$, $y_1 = y_0 + h(2x_0 y_0^2) = 1 + (0.05)(2(0)(1)^2) = \underline{\underline{1}}$

$x_2 = 0.10$, $y_2 = y_1 + h(2x_1 y_1^2) = 1 + 0.05(2(0.05)(1)^2)$

$= 1 + 0.005 = \underline{\underline{1.005}}$

$x_3 = 0.15$, $y_3 = y_2 + h(2x_2 y_2^2) = 1.005 + (0.05)(2(0.1)(1.005)^2)$

$= 1.005 + 0.0101003 = \underline{\underline{1.0151003}}$

From my Excel worksheet: $y(0.4) \approx y_8 = \underline{\underline{1.157878175}}$

D) (con't)

Stepsize $h = 0.02$

$x_0 = 0$, $y_0 = 1$

$x_1 = 0.02$, $y_1 = y_0 + h(2x_0y_0^2) = 1 + 0.02(2(0)(1)^2)$
$= \underline{\underline{1}}$

$x_2 = 0.04$, $y_2 = y_1 + h(2x_1y_1^2) = 1 + (0.02)(2(0.02)(1)^2)$
$= 1 + 0.0008 = \underline{\underline{1.0008}}$

$x_3 = 0.06$, $y_3 = y_2 + h(2x_2y_2^2) = 1.0008 + (0.02)(2(0.04)(1.0008)^2)$
$= 1.0008 + 0.0016026$
$= \underline{\underline{1.0024026}}\ldots$

From my Excel worksheet, $y(0.4) \approx y_{20} = \underline{\underline{1.176875792\ldots}}$

Stepsize $h=0.01$:

$x_0 = 0$, $y_0 = 1$

$x_1 = 0.01$, $y_1 = y_0 + h(2x_0y_0^2) = 1 + 0.01(2(0)(1)^2) = \underline{1}$

$x_2 = 0.02$, $y_2 = y_1 + h(2x_1y_1^2) = 1 + (0.01)(2(0.01)(1)^2)$
$= 1 + 0.0002 = \underline{\underline{1.0002}}$

$x_3 = 0.03$, $y_3 = y_2 + h(2x_2y_2^2) = 1.0002 + (0.01)(2(0.02)(1.0002)^2)$
$= 1.0002 + 0.0004002$
$= \underline{\underline{1.0006002}}\ldots$

From my Excel worksheet, $y(0.4) \approx y_{40} = \underline{\underline{1.183573746\ldots}}$

1) (con't)

Answers will vary. My thoughts:

The value estimated using a stepsize of $h = 0.01$ should be closer to the actual value of $y(0.4)$ than the value estimated using a stepsize of $h = 0.1$. Reducing the stepsize allows us to travel a shorter distance along the slope field before stopping to find the current direction of the slope field of the solution y. Two practical limitations to reducing the stepsize indefinitely are computational time (which will increase as stepsize decreases) and roundoff error (which accumulates with each step).

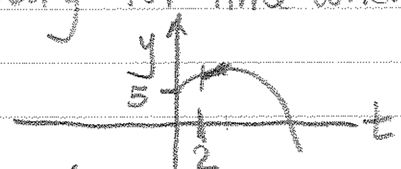
Solns can diverge w/ b/c of R.O. error

2) $v(0) = 42$ ft/s initial velocity of ball

$y(t) = 5 + 42t - 16t^2$ height of ball in feet after t seconds

a) Find the average velocity for time when $t = 2$ and lasting

i) 1/2 second



$$\text{avg vel} = \frac{\Delta y}{\Delta t} = \frac{y(2.5) - y(2)}{(2.5s) - (2s)} = \frac{\left(5 + 42(2.5) - 16(2.5)^2\right) - \left(5 + 42(2) - 16(2)^2\right)}{0.5s}$$

$$= \frac{10ft - 25ft}{0.5s} = \underline{-30 \text{ ft/s}}$$

ii) 1/10 second

$$\text{Avg vel} = \frac{\Delta y}{\Delta t} = \frac{y(2.1) - y(2)}{2.1 - 2\text{sec}} = \frac{\left(5 + 42(2.1) - 16(2.1)^2\right) - 25}{0.1s}$$

$$= \frac{22.64 \text{ ft} - 25 \text{ ft}}{0.1s} = \underline{-23.6 \text{ ft/s}}$$

2) (con't)

a) iii) 1/100 second

$$\text{Avg Vel} = \frac{\Delta y}{\Delta t} = \frac{y(2.01) - y(2)}{(2.01 - 2)\text{ sec}} = \frac{(5 + 42(2.01) - 16(2.01)^2) - 25\text{ ft}}{0.01\text{ sec}}$$

$$= \frac{24.7784\text{ ft} - 25\text{ ft}}{0.01\text{ sec}} = \underline{\underline{-22.16\text{ ft/s}}}$$

b) Estimate the instantaneous velocity when $t = 2$:

I'll consider shorter & shorter timesteps:

$$\text{Avg vel} = \frac{\Delta y}{\Delta t} = \frac{y(2.001) - y(2)}{0.001\text{ s}} = -22.016\text{ ft/s}$$

$$\text{Avg vel} = \frac{\Delta y}{\Delta t} = \frac{y(2.0001) - y(2)}{0.0001\text{ s}} = -22.0016\text{ ft/s}$$

$$\text{Avg vel} = \frac{\Delta y}{\Delta t} = \frac{y(2.00001) - y(2)}{0.00001\text{ s}} = -22.0001\ldots\text{ ft/s}$$

The instantaneous velocity at $t = 2$ appears to be " $\underline{\underline{-22\text{ ft/s}}}$. It is the limit of average velocities.

3) [Answers May Vary]