EASY	MEDIUM	HARDER
1-4	5 9	10(#34)
6	7 11	10(#27)
12	8	10(#30)

Sample Solutions

①

Problem Solving Lab, Lesson 46 Board Problems

1) $\nabla f(x,y) = \langle f_x, f_y \rangle = \langle 2x, 2y \rangle$

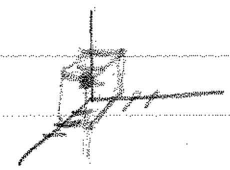
$\nabla f(1,2) = \langle 2, 4 \rangle$

$\underset{\sim}{v} = \langle 2, -3 \rangle \Rightarrow \underset{\sim}{u} = \frac{\underset{\sim}{v}}{\|\underset{\sim}{v}\|} = \frac{\langle 2, -3 \rangle}{\sqrt{4+9}} = \left\langle \frac{2}{\sqrt{13}}, \frac{-3}{\sqrt{13}} \right\rangle$

$D_{\underset{\sim}{u}} f(1,2) = \langle 2, 4 \rangle \cdot \left\langle \frac{2}{\sqrt{13}}, \frac{-3}{\sqrt{13}} \right\rangle = \frac{4}{\sqrt{13}} - \frac{12}{\sqrt{13}} = \frac{-8}{\sqrt{13}}$

2) $\nabla f(x,y,z) = \langle 2x, 2y, 2z \rangle$

$\nabla f(2,1,3) = \langle 4, 2, 6 \rangle$



$\underset{\sim}{v} = \langle 0,0,0 \rangle - \langle 2,1,3 \rangle = \langle -2,-1,-3 \rangle$

$\|\underset{\sim}{v}\| = \sqrt{4+1+9} = \sqrt{14}$

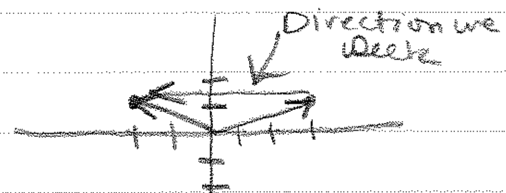
$D_{\underset{\sim}{u}} f(2,1,3) = \nabla f(2,1,3) \cdot \underset{\sim}{u} = \langle 4,2,6 \rangle \cdot \left\langle \frac{-2}{\sqrt{14}}, \frac{-1}{\sqrt{14}}, \frac{-3}{\sqrt{14}} \right\rangle$

$= \frac{-8}{\sqrt{14}} - \frac{2}{\sqrt{14}} - \frac{18}{\sqrt{14}} = \frac{-28}{\sqrt{14}}$

3) $\nabla f(x,y) = \langle 3(x-y)^2 + 4x^3, 3(x-y)^2(-1) \rangle$

$\nabla f(3,1) = \langle 3(2)^2 + 4(27), 3(2)^2(-1) \rangle$

$\nabla f(3,1) = \langle 120, -12 \rangle$



$\underset{\sim}{v} = \langle -2, 1 \rangle - \langle 3, 1 \rangle = \langle -5, 0 \rangle$

$\|\underset{\sim}{v}\| = 5$

$D_{\underset{\sim}{u}} f(3,1) = \langle 120, -12 \rangle \cdot \langle \frac{-5}{5}, 0 \rangle = -120$

4) $f_x = \frac{2}{y+z}$ $\quad f_y = 2x(-1(y+z)^{-2})$ $\quad f_z = 2x(-(y+z)^{-2})$

$\nabla f(4,1,1) = \left\langle \frac{2}{1+1}, \frac{-2(4)}{(1+1)^2}, \frac{-2(4)}{(1+1)^2} \right\rangle = \left\langle 1, \frac{-8}{4}, \frac{-8}{4} \right\rangle$

$\nabla f(4,1,1) = \langle 1, -2, -2 \rangle$

$\|\underset{\sim}{v}\| = \sqrt{1+4+9} = \sqrt{14}$

$D_{\underset{\sim}{u}} f(4,1,1) = \langle 1,-2,-2 \rangle \cdot \left\langle \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right\rangle = \frac{1}{\sqrt{14}} - \frac{4}{\sqrt{14}} - \frac{6}{\sqrt{14}} = \frac{-9}{\sqrt{14}}$

Sample Solutions (2)

PSL, Lesson 46 Board Problems

5) $f(x,y,z) = x^5 + 3x^4y^4z^3 + yz^2$

$f_x = 5x^4 + 4(3y^4z^3)x^3 + 0$

$= 5x^4 + 12x^3y^4z^3$

$f_{xy} = 0 + 4(12x^3z^3)y^3$

$= 48x^3y^3z^3$

$f_{xyz} = 3(48x^3y^3)z^2 = 144x^3y^3z^2$

6) $V(r,h) = \pi r^2 h$

$\frac{\partial V}{\partial r} = 2\pi r h$ This how much the volume of a cylinder is changing if its height stays fixed but its radius changes.

7) $f(x,y) = 5xy - 2y^2$

$\nabla f(x,y) = \langle 5y, 5x - 10y \rangle$

The rate of change of $f(x,y)$ is at a maximum in the direction of the gradient. So, at point $(1,3)$ the direction is $\langle 15, -25 \rangle$.

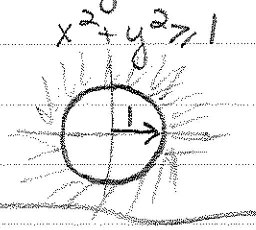
[For more details on "why" see your lecture notes on $D_u f = \nabla f \cdot \underline{u} = \|\nabla f\| \|\underline{u}\| \cos\theta_{\nabla f, u}$]

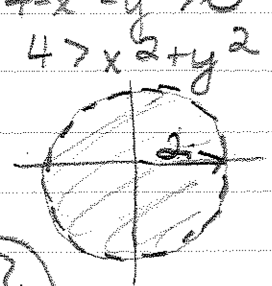
8) Find AND sketch the domain of

$f(x,y) = \sqrt{x^2+y^2-1} + \ln(4-x^2-y^2)$.

Restrictions: $x^2+y^2-1 \geq 0$ AND $4-x^2-y^2 > 0$

$x^2+y^2 \geq 1$ $\quad$ $4 > x^2+y^2$





$D = \{(x,y) : 1 \leq x^2+y^2 < 4\}$

Sample Solution (3)

PSL, Lesson #6, Board Problems

9) $D_{\underset{\sim}{u}} f(3,-1) = \nabla f(3,-1) \cdot \underset{\sim}{u} = 0$ ← unknown

$\nabla f = \langle 2xy^2+3 \, , \, 2x^2y \rangle$

$\nabla f(3,-1) = \langle 2(3)(-1)^2+3, \, 2(9)(-1) \rangle = \langle 9, -18 \rangle$

$\underset{\sim}{u} = \langle u_1, u_2 \rangle$ and $\sqrt{u_1^2+u_2^2} = 1$

$0 = \langle 9, -18 \rangle \cdot \langle u_1, u_2 \rangle = 9u_1 - 18u_2$

$\Rightarrow u_1 = 2u_2$. Since $u_1^2+u_2^2=1$, it means $u_2 = \pm\frac{1}{\sqrt{5}}$.

All the directions will be multiples of $\langle 2, 1 \rangle$, i.e.

$\underset{\sim}{u} = \left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle$ or any constant times $\underset{\sim}{u}$.

10) Sec 11.6 #30

$z = 1000 - 0.005x^2 - 0.01y^2$ starting at point $(60, 40, 966)$

(a) $D_{\underset{\sim}{u}} f(60,40) = \langle -0.010(60), -0.02(40) \rangle \cdot \langle 0, -1 \rangle = 0.8$

← $\underset{\sim}{u} = \langle 0, -1 \rangle$

i.e. vertical meters per horizontal meters

You will ascend at rate of 0.8 m per m.

(b) ↖ Northwest corresponds to $\underset{\sim}{v} = \langle -1, 1 \rangle \Rightarrow \underset{\sim}{u} = \left\langle \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$

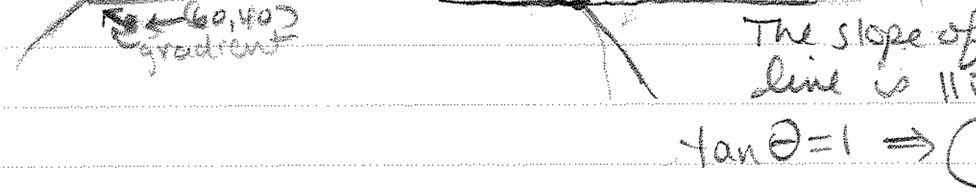
$D_{\underset{\sim}{u}} f(60,40) = \langle -0.01(60), -0.02(40) \rangle \cdot \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$

$\frac{.6}{\sqrt{2}} - \frac{.8}{\sqrt{2}} = \frac{-0.2}{\sqrt{2}} = -0.1412\ldots$

You will descend at 0.14 m per meter.

(c) The slope is largest in direction of gradient:

$\nabla f(60,40) = \langle 0.6, -0.8 \rangle$. The rate of ascent will be $\|\nabla f(60,40)\| = 1$, i.e. 1 m per m.



The slope of the tangent line is $\|\nabla f\| = \underline{1}$, so

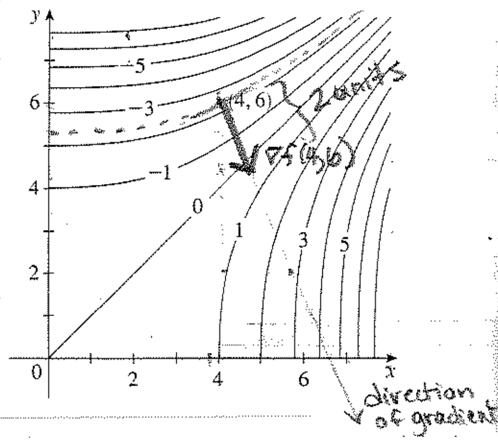
$\tan\theta = 1 \Rightarrow \theta = 45°$

PSL lesson 46, Sample Solutions (con't)

10) (con't) Sec 11.6 #34

$-3 \le f(4,6) \le -2$, so $f(4,6) \approx -2.5$. The direction of the gradient should be orthogonal to the level curve $f(x,y) = -2.5$, drawn as a dotted line. The tangent to this curve at $(4,6)$ has slope $\approx 3/8$, so the slope of the line perpendicular to it should be $-8/3$. The gradient should have direction $\langle 3,-8\rangle$. To figure out the magnitude, estimate it using the rate of change at point $(4,6)$ in the direction $\langle 3,-8\rangle$. Looking at the 2 closest level curves, i.e., $f(x,y) = -2$ and $f(x,y) = -3$, I calculate the distance between them to be ≈ 0.5 units. Therefore, the rate of change will be $\frac{-2-(-3)}{0.5} = \frac{1}{1/2} = 2$ units.

34. Sketch the gradient vector $\nabla f(4, 6)$ for the function f whose level curves are shown. Explain how you chose the direction and length of this vector.



Sec 11.6 # 27

$$T(x,y,z) = \frac{k}{\sqrt{(x-0)^2+(y-0)^2+(z-0)^2}} = \frac{k}{\sqrt{x^2+y^2+z^2}}$$

$$T(1,2,2) = \frac{k}{\sqrt{1+4+4}} = 120^\circ \Rightarrow k = 360^\circ$$

$$\therefore\ T(x,y,z) = \frac{360^\circ}{\sqrt{x^2+y^2+z^2}}$$

a) $\underset{\sim}{v} = \langle 2,1,3\rangle - \langle 1,2,2\rangle = \langle 1,-1,1\rangle$

$$\underset{\sim}{u} = \frac{\langle 1,-1,1\rangle}{\sqrt{1^2+(-1)^2+1^2}} = \left\langle \frac{1}{\sqrt3}, \frac{-1}{\sqrt3}, \frac{1}{\sqrt3}\right\rangle$$

$$T_x = -\tfrac12(360^\circ)(x^2+y^2+z^2)^{-3/2}(2x) = \frac{-360^\circ x}{(x^2+y^2+z^2)^{3/2}}$$

$$T_y = -\tfrac12(360^\circ)(x^2+y^2+z^2)^{-3/2}(2y) = \frac{-360^\circ y}{(x^2+y^2+z^2)^{3/2}}$$

$$T_z = -\tfrac12(360^\circ)(x^2+y^2+z^2)^{-3/2}(2z) \qquad \frac{-360^\circ z}{(x^2+y^2+z^2)^{3/2}}$$

$$T_x(1,2,2) = -40/3 = -13.\overline{3}$$

$$T_y(1,2,2) = -80/3 = -26.\overline{6}$$

$$T_z(1,2,2) = -80/3 = -26.\overline{6}$$

$$D_{\underset{\sim}{u}} f(1,2,2) = \nabla f(1,2,2) \cdot \underset{\sim}{u} = \left\langle -40/3, -80/3, -80/3\right\rangle \cdot \left\langle \frac{1}{\sqrt3}, \frac{-1}{\sqrt3}, \frac{1}{\sqrt3}\right\rangle$$

$$= \frac{-40}{3\sqrt3} + \frac{80}{3\sqrt3} - \frac{80}{3\sqrt3} = \frac{-40}{3\sqrt3}$$

degrees per 1 distance unit.

PSL Lesson 46, Sample Solutions (con't)

10 (con't) Sec 11.6 #27 (con't)

b) The direction of greatest increase is the direction of the gradient. At any point, the gradient is

$$\nabla T(x,y,z) = \left\langle \frac{-360^\circ x}{(x^2+y^2+z^2)^{3/2}}, \frac{-360^\circ y}{(x^2+y^2+z^2)^{3/2}}, \frac{-360^\circ z}{(x^2+y^2+z^2)^{3/2}} \right\rangle$$

For any point away from the origin, this points to the origin. To see that more clearly, consider $\frac{\nabla T}{\|\nabla T\|}$.

$$\|\nabla T(x,y,z)\| = \sqrt{\left(\frac{-360^\circ x}{(x^2+y^2+z^2)^{3/2}}\right)^2 + \left(\frac{-360^\circ y}{(x^2+y^2+z^2)^{3/2}}\right)^2 + \left(\frac{-360^\circ z}{(x^2+y^2+z^2)^{3/2}}\right)^2}$$

$$= \sqrt{\frac{(360^\circ)^2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^3}} = \frac{360^\circ}{x^2+y^2+z^2}$$

So $\frac{\nabla T}{\|\nabla T\|} =$

$$\left\langle \frac{\frac{-360^\circ x}{(x^2+y^2+z^2)^{3/2}}}{\frac{360^\circ}{x^2+y^2+z^2}}, \frac{\frac{-360^\circ y}{(x^2+y^2+z^2)^{3/2}}}{\frac{360^\circ}{x^2+y^2+z^2}}, \frac{\frac{-360^\circ z}{(x^2+y^2+z^2)^{3/2}}}{\frac{360^\circ}{x^2+y^2+z^2}} \right\rangle$$

$$= \left\langle \frac{(-360^\circ x)(x^2+y^2+z^2)}{(360^\circ)(x^2+y^2+z^2)^{3/2}}, \frac{(-360^\circ y)(x^2+y^2+z^2)}{(360^\circ)(x^2+y^2+z^2)^{3/2}}, \frac{(-360^\circ z)(x^2+y^2+z^2)}{(360^\circ)(x^2+y^2+z^2)^{3/2}} \right\rangle$$

$$= \left\langle \frac{-x}{x^2+y^2+z^2}, \frac{-y}{x^2+y^2+z^2}, \frac{-z}{x^2+y^2+z^2} \right\rangle$$

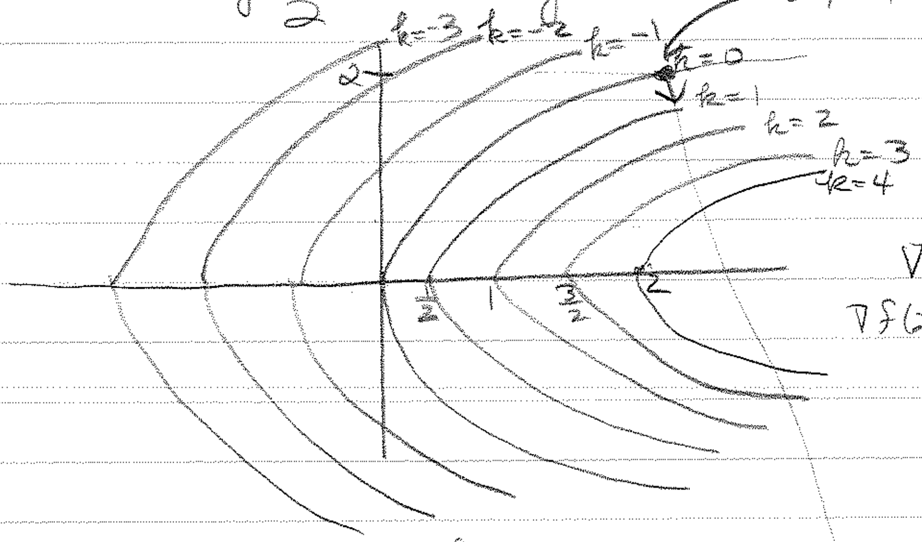
$$= \frac{1}{x^2+y^2+z^2} \underbrace{\langle -x, -y, -z \rangle}$$

For any point (x,y,z), this vector points to the origin.

PSL lesson 46, Sample Solutions (con't)

11) $2x - y^2 = k$

$x = \dfrac{y^2 + k}{2} = \frac{1}{2}y^2 + \frac{1}{2}k$



$2 = \frac{1}{2}(2)^2 + \frac{1}{2}k$

$2 \cdot 2 = 2^2 + k$

$4 = 4 + k$

$0 = k$

$\nabla f = \langle 2, -2y \rangle$

$\nabla f(2,2) = \langle 2, -4 \rangle$

Gradient vector is perpendicular to level curves.

Mathematica: ContourPlot[2x-y^2, {x,-2,2}, {y,-2,2}, ContourShading→False, Contours={-3,-2,-1,0,1,2,3,4}]

12) $\dfrac{\partial g}{\partial r} = \dfrac{1}{3r^2 s} \cdot 6r - \cos(3rt)(3t) = \dfrac{2}{rs} - 3t\cos(3rt)$

$\dfrac{\partial g}{\partial s} = \dfrac{1}{3r^2 s} \cdot 3r^2 = \dfrac{1}{s}$

$\dfrac{\partial g}{\partial t} = -\sin(5t)5 - \cos(3rt)(3r) = -5\sin(5t) - 3r\cos(3rt)$

$\dfrac{\partial g}{\partial r}(\overset{r}{1}, \overset{s}{2}, \overset{t}{2/10}) = \dfrac{2}{(1)(2)} - 3\left(\frac{2}{10}\right)\cos\left(3(1)\frac{2}{10}\right) = 1 - \frac{3}{5}\cos\left(\frac{3}{5}\right)$

$\dfrac{\partial g}{\partial s}(1, 2, 2/10) = \dfrac{1}{2}$

$\dfrac{\partial g}{\partial t}(1, 2, 2/10) = -5\sin(1) - 3(1)\cos\left(3(1)\left(\frac{2}{10}\right)\right) = -5\sin(1) - 3\cos\left(\frac{3}{5}\right)$